Quantum Chaos : Learning from network models

Quantum graphs, weak localisation and RMT

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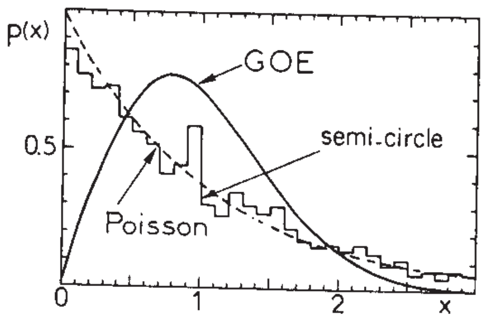
[1] Quantum Chaos and RMT

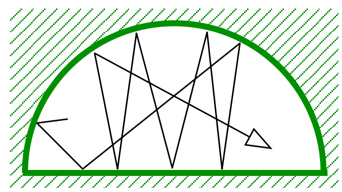
Bohigas-Gianonni-Schmit (BGS) Conjecture ’84 :

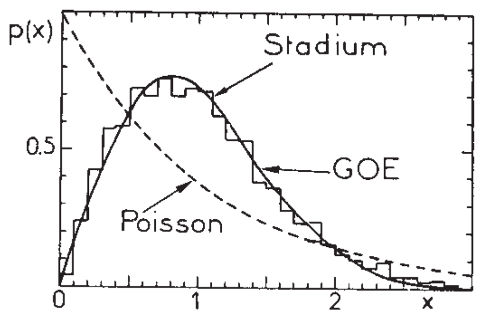
Quantum systems with chaotic classical dynamics have random matrices level-statistics.

time-reversal symmetry $\Rightarrow$ GOE level-statistics

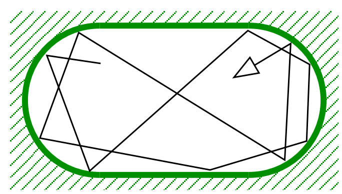


Integrable





Chaotic



Form factor $K(\tau) =$ FT of level-level correlation funct.

$$K_{\mathrm{RMT}}(\tau) = 2\tau - \tau \log(1 + 2\tau)$$
$$= 2\tau - 2\tau^2 + 2\tau^3 + \mathcal{O}[\tau^4]$$

Objective : Find $K(\tau)$ for chaotic system thereby prove or disprove BGS-conjecture

Hard problem — forced to work with toy models

[2] Periodic orbit theory

Gutzwiller '71 : ray-optics limit of quantum mechanics

Feynman path integral for small $\hbar$

$\longrightarrow$ Sum over classical orbits P & Q

$$K(\tau) = \left\langle \frac{1}{\Delta} \sum_{P,Q} A_P A_Q^* \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S_P - S_Q)} \right\rangle$$

where $\langle \cdots \rangle$ indicates Energy Averaging

Δ is mean level spacing

and $\tau = t\Delta/\hbar$

Berry diagonal approximation '85 : $Q = P$

strictly it is any cyclic permutation of P

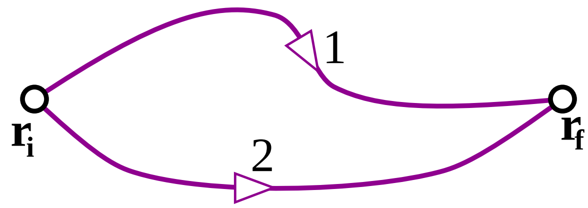
$$K(\tau) = 2\tau + \cdots$$

No systematic correction scheme

- **Corrections within periodic orbit theory : YES**
- **Corrections beyond periodic orbit theory : ?**

[3] Diag. approx and weak localisation

Semiclassical limit ($S_\alpha \gg \hbar$)



$$P(\mathbf{r}_\mathrm{i} \to \mathbf{r}_\mathrm{f}) = \left| A_1 \mathrm{e}^{\mathrm{i}S_1/\hbar} + A_2 \mathrm{e}^{\mathrm{i}S_2/\hbar} \right|^2$$
$$= A_1^2 + A_2^2 + 2A_1 A_2 \cos\left[(S_1 - S_2)/\hbar\right]$$

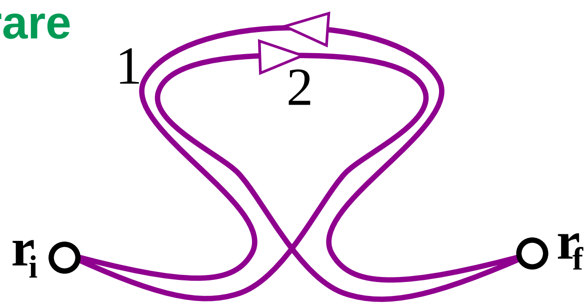
Diag. Approx : IGNORE correlations between S_1 and S_2

then with averaging :

$$P(\mathbf{r}_\mathrm{i} \to \mathbf{r}_\mathrm{f}) \simeq A_1^2 + A_2^2$$

Correlations may be rare

but they exist

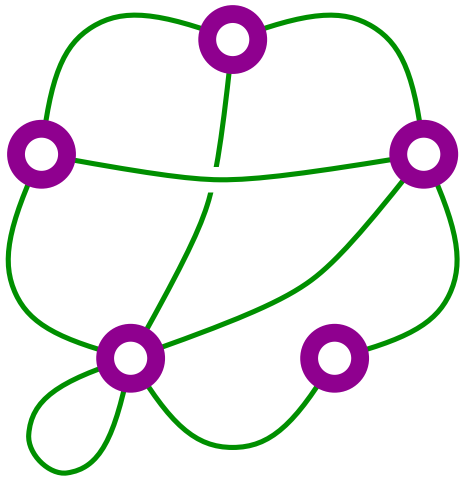


Paths of this type

$\rightarrow$ **weak localisation in disordered systems**

[4] Quantum Graphs

Kottos-Smilansky '99

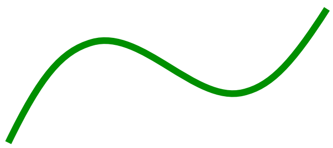


Periodic orbit theory is exact

Numerics show GOE level-statistics

B Bonds
N Nodes

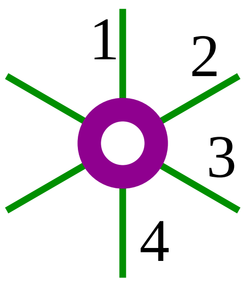
Elements of quantum graph

(1) 

BONDS : 1-D wave guide

$$\psi_i(x,t) = \mathrm{e}^{\mathrm{i}p(x-x_0)+\mathrm{i}Et}\psi_i(x_0,0)$$

Lengths of all bonds are incommensurate

(2) 

NODES : Unitary Scatterers

$$\psi_{i,\mathrm{out}} = \sigma_{ij}\psi_{j,\mathrm{in}}$$

Unitarity : $\sum_j \sigma_{ij}\sigma^*_{jk} = \delta_{ik}$

Form factor $\Rightarrow$ **double sum over periodic orbits**

$$K(\tau) = \frac{1}{B}\sum_{P,Q} A_P A_Q^* \delta_{L_P,L_Q}$$

(sum over all cyclic permutations of each orbit)

[5] Diagonal approximation and beyond

Diagonal approximation for graphs : $Q = P$

$$\sum_{P,Q} A_P A_Q^* \delta_{L_P, L_Q} \longrightarrow t \sum_P |A_P|^2$$

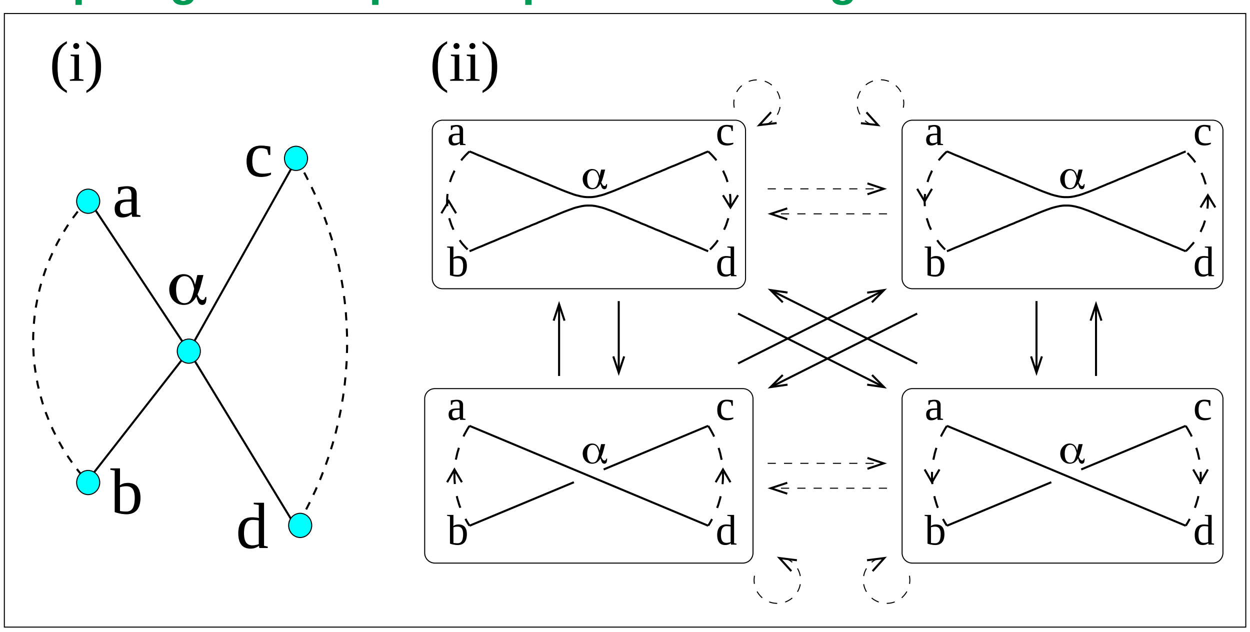
...and BEYOND

Corrections to diag. approx. — weak localisation

P and Q identical everywhere except at intersections.

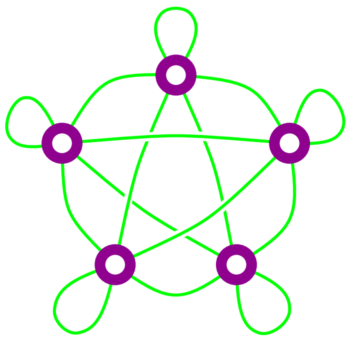
Do systematic expansion in number of intersections.

Topologies of a pair of paths with single intersection



[6] Naive estimate

Estimate of leading correction to diag. approx. for the following simple graph



maximally-connected graph $B = \frac{1}{2}N^2$

with "s-wave" nodes: $|\sigma_{ab}^{(\alpha)}| = 1/N$

Classical weight of single-intersection compared to zero-intersections $= t^2 \times \frac{1}{N}$

 # ways to place intersection (→ t^2)

One node at intersection constrained (→ $\frac{1}{N}$)

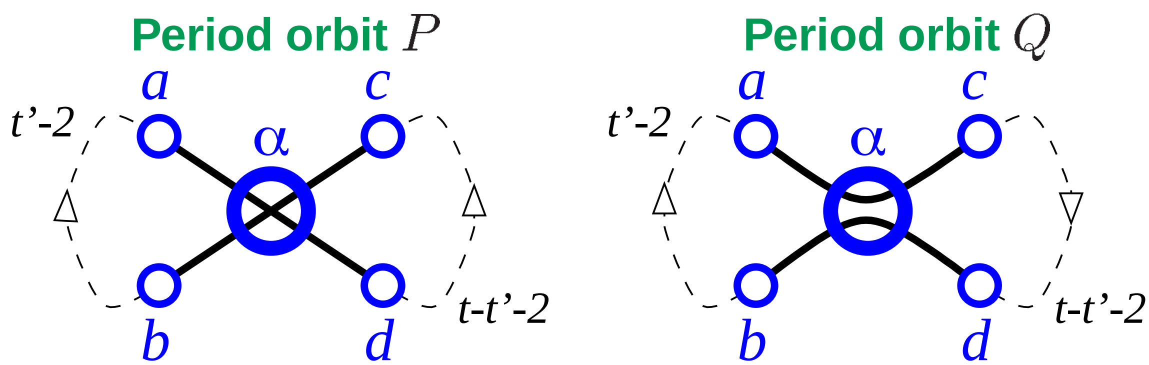
Naive estimate of their contribution to form factor

$$K^{(1)}_{\text{estimate}}(\tau) \sim \frac{t}{B} \times \left(t^2 \times \frac{1}{N} \right) = N^3 \tau^3$$

This naive estimate is very WRONG !!

- **Not a τ^2-correction**
- **For $\tau \sim 1$ and $N \to \infty$ then $K^{(1)}_{\text{estimate}}(\tau) \to \infty$**

[7] Basics of calculation



$$K_1(\tau) = \frac{t^2}{B} \sum_{t'=3}^{t-3} \sum_{\alpha,a,b,c,d} M^{t'-1}_{\alpha\leftarrow a, b\leftarrow\alpha} M^{t-t'-1}_{\alpha\leftarrow c, d\leftarrow\alpha}$$

$$\times\ \sigma^{(\alpha)}_{da} \sigma^{(\alpha)}_{bc} \sigma^{(\alpha)*}_{ca} \sigma^{(\alpha)*}_{db}$$

Assume Graph is ergodic $\Rightarrow \lim_{t\to\infty} M^t_{m'\leftarrow l', m\leftarrow l} = \frac{1}{B}$

- $t \to \infty$ means $t \gg$ classical ergodic time
- Either t' or $t - t'$ satisfy this limit.

$\Rightarrow$ above sum can be evaluated

...still get WRONG answer [result $\sim N^2\tau^3$]
unless careful about special cases

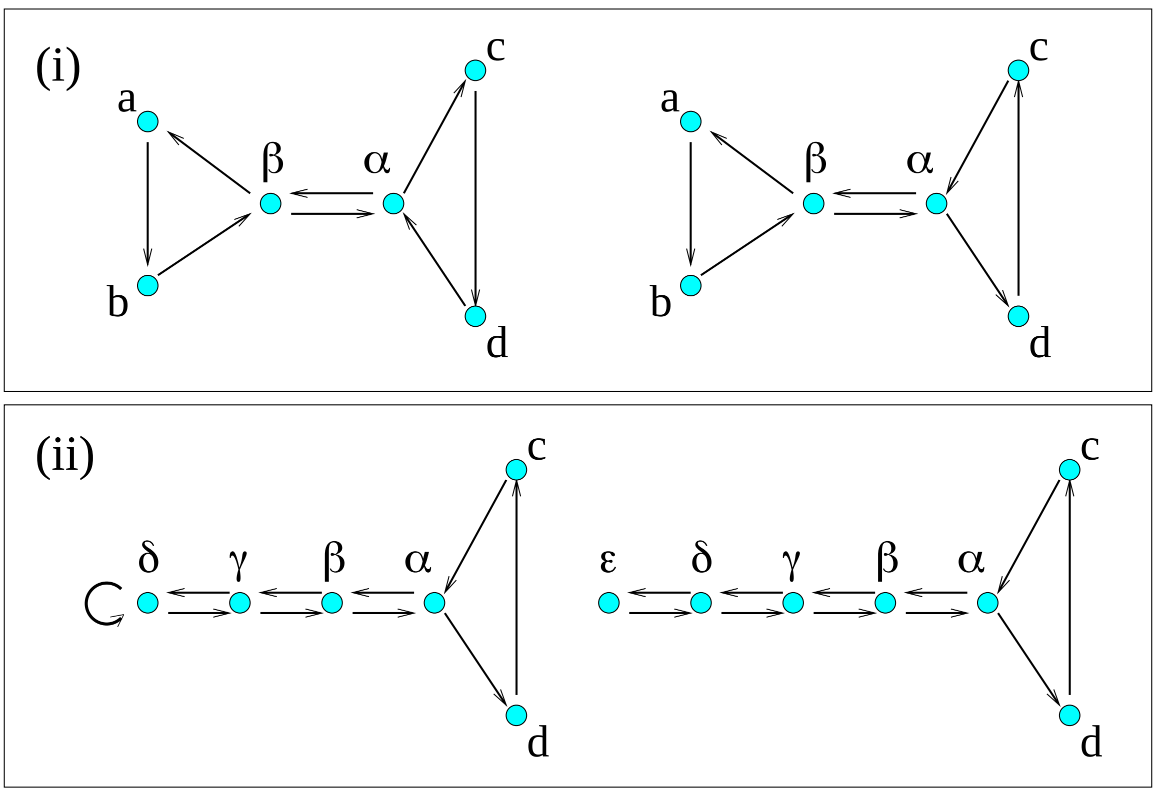
[8] Special cases and exceptions

(i) Must not double-count orbits of the form:

$$P = \alpha \to c \to \cdots \to d \to \alpha \to \beta \to a \to \cdots \to b \to \beta \to \alpha$$

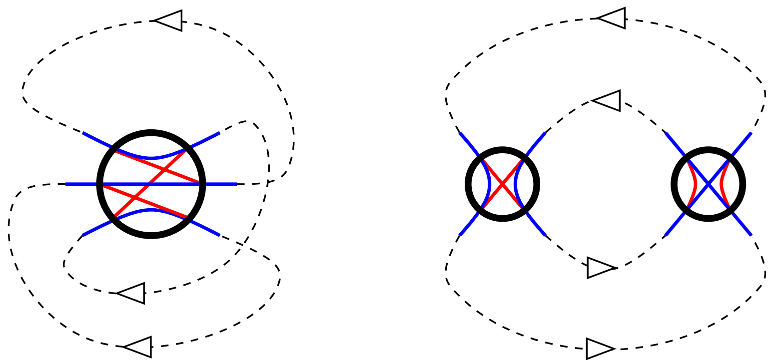
$$Q = \alpha \to d \to \cdots \to c \to \alpha \to \beta \to a \to \cdots \to b \to \beta \to \alpha$$

(ii) Must not count orbits where one loop self-retraces.



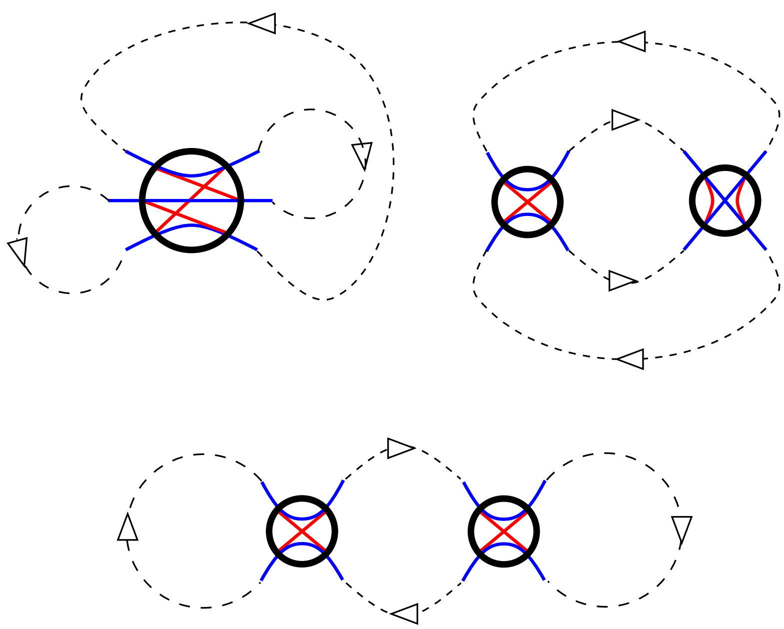
[9] Order τ^3 contributions

- **Without time-reversal symmetry**



- **With time-reversal symmetry**

Above diagrams plus



[10] Summary

- Periodic orbit theory on quantum graphs
- Calculate form factor : $K(\tau)$
  (Fourier trans. of level-level correlation function)
- We go beyond the diagonal approximation

- Expansion in # of intersections of path with itself.
  Analogous to weak localisation in disordered syst.
- We show mixing (ergodic) graph has

$$K(\tau) = \begin{cases} 2\tau - 2\tau^2 + \cdots \\ \quad \tau + \quad 0 \quad + \; 0 + \cdots \end{cases}$$

diag. approx.
beyond diag. approx.
"systematic" correction scheme

- Agreement with RMT
  beyond the diagonal approximation