# Quantum to classical crossover in chaotic transport

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very useful discussions: E.V. Sukhorukov

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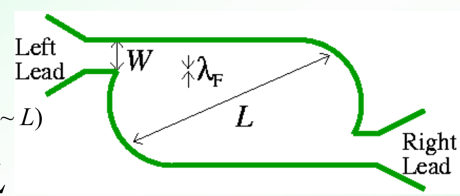
## Overview

**System :** open chaotic cavity

**clean** or smooth disorder (correlation length ~ $L$)




$$\lambda_F << W << L$$

semiclassical — classically chaotic

♣ **Transmission eigenvalues** "all" 0 or 1 in classical limit

Shot noise : **fano factor** = (RMT result) × $\exp[-t_E/\tau_D]$

♣ **weak localisation**

correction to conductance = (RMT result)

no factor of $\exp[-t_E/\tau_D]$ ; fits recent numerics
contradicting earlier theories?

Aleiner-Larkin (1996), Adagideli(2003)

♣ Phenomenological **"two phase fluid" model**

Silverstrov-Goordon-Beenakker (2003)

# When does RMT behaviour break down?

**Quantum chaos** is typically
"well–explained" by random matrix theory (RMT)

**closed cavity :** level–statistics

**open cavity :**
- ♠ weak localisation (magnetoconductance) = -1/4
- ♠ universal conductance fluctuations = 1/8
- ♠ shot noise : Fano factor = 1/4

**STILL NEED** microscopic justification for this RMT–behaviour.

huge recent progress : Muller-Heusler-Braun-Haake-Altland(2004)
Muller-Altland(2004)

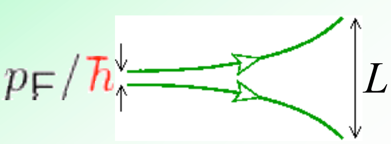
## ...when the Ehrenfest time becomes relevant.

then behaviour ceases to be RMT Aleiner-Larkin (1996)

# The Ehrenfest time, $t_{\mathrm{E}}$ .

Ehrenfest time = time for **minimal wavepacket** to spread
over system's phase–space

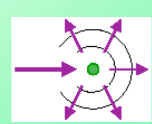
When potential is smooth on scale of wavelength:
Wavepacket spreads under **classical chaotic flow**




$$t_{\mathrm{E}} = \lambda^{-1} \ln[p_{\mathrm{F}} L/\hbar]$$

Lypunov exponent

Cf. Potential with $\delta$–correlated disorder:



wavepacket covers momentum–space after $\tau$
and diffuses in position–space ➡ $t_{\mathrm{E}} \approx t_{\mathrm{Thouless}} \sim (L\tau/v_{\mathrm{F}})^{1/2}$

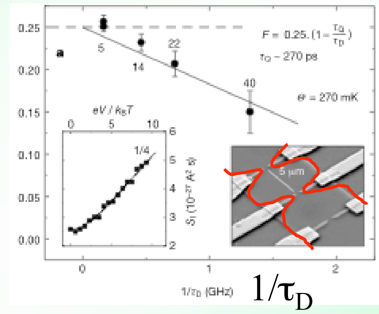
## Experiments (real & numerical)

**Fano–factor in real device**

Oberholzeer-Sukhorukov-Schonenberger(2002)

**they observe** $F \sim 1/4 \times (1- t_E/\tau_D)$

$\sim 1/4 \times \exp[- t_E/\tau_D]$



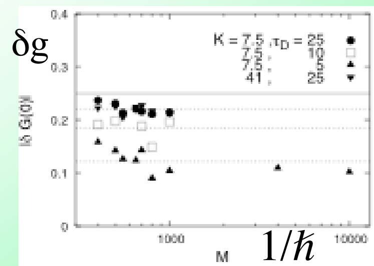

**Fano–factor & weak–localisation in numerics**

Tworzydlo-Tajic-Beenakker (2004)

**they observe** $F \sim 1/4 \times (1- t_E/\tau_D)$

**But** $\delta g \sim$ independent of $t_E$

**Model = kicked rotator with "leads"**




## Transmission eigenvalues

**Scattering matrix :** $\mathrm{S} = \begin{pmatrix} \mathbf{r} & \mathbf{t}^\dagger \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}$

**Transmission matrix :** $\mathrm{T} \equiv \mathrm{t}^\dagger \mathrm{t}$

**Conductance & current noise (Landauer-Buttiker)**

**dimensionless conductance :** $g = \mathrm{tr}[\mathrm{T}] = \sum_n T_n$

**weak-localisation : non–trivial contribution to t.**

**Shot noise :** $S = \int \mathrm{d}t \langle I(t)I(0) - I^2 \rangle \propto \mathrm{tr}[\mathrm{T}(1-\mathrm{T})]$

**Fano factor** $= S/S_{\text{Poisson}} = \dfrac{\mathrm{tr}[\mathrm{T}(\mathrm{T}-1)]}{\mathrm{tr}[\mathrm{T}]} = \dfrac{\sum_n T_n(1-T_n)}{\sum_n T_n}$

**....more physically useful; Fano factor** $\propto S / I$

# Conjectures we aim to prove

**(A) Classical-limit conjecture** Beenakker-van Houten (1991)

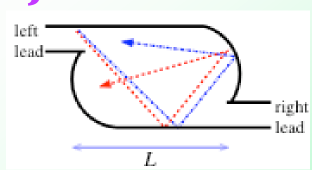
Numerical observation : $T_n \in \{0,1\}$ ➡ **noise-less** transmission

**Each classical path** is perfectly transmitted $T_\alpha = 1$
or perfectly reflected $T_\alpha = 0$

**Conjecture :** Classical paths $\xrightarrow{\text{Leap of faith}}$ transmission modes

**(B) Two-phase-fluid conjecture** Silverstrov-Goordon-Beenakker (2003)

Numerical observation :

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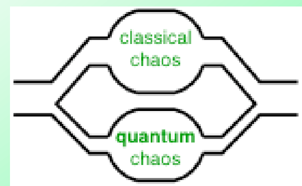
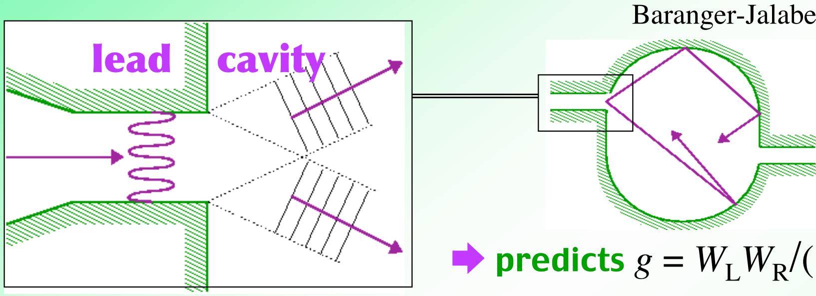



**classical** chaos = classical limit in **(A)** above
**quantum** chaos ≈ random matrix theory (RMT)

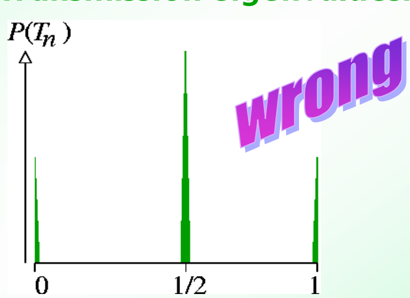
# Problem with the "leap of faith"

Baranger-Jalabert-Stone (1993)



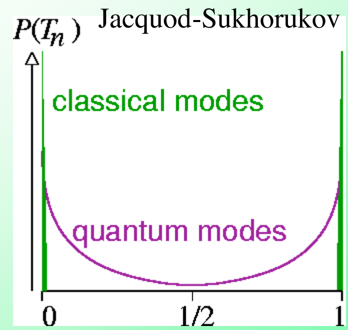

➡ predicts $g = W_L W_R / (W_L + W_R)$ ✓

➡ predicts distrib. of
Transmission eigenvalues:



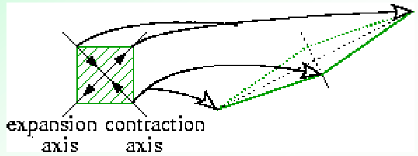

**Numerical observation**

Jacquod-Sukhorukov (2004)

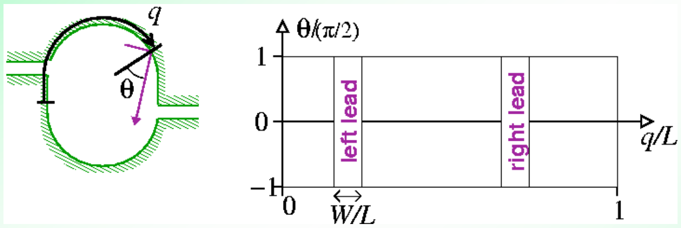


# Classical phase-space to Birkoff map

**Louvillian flow**
in classical phase-space



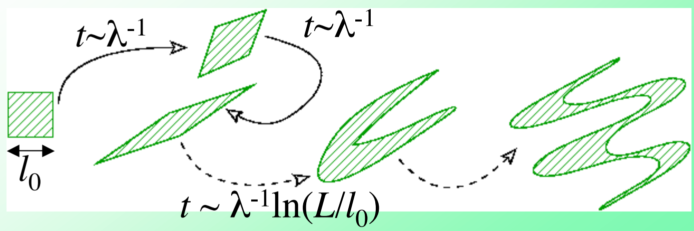

**Poincare surface of section** for boundary of cavity



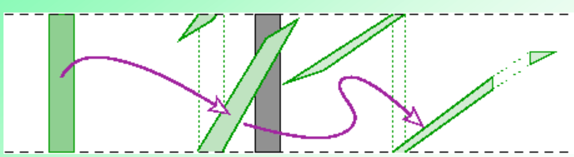

**Birkoff map**

$$\theta_{i+1} = f(\theta_i, q_i)$$
$$q_{i+1} = g(\theta_i, q_i)$$

**Louvillian flow**
on surface of section



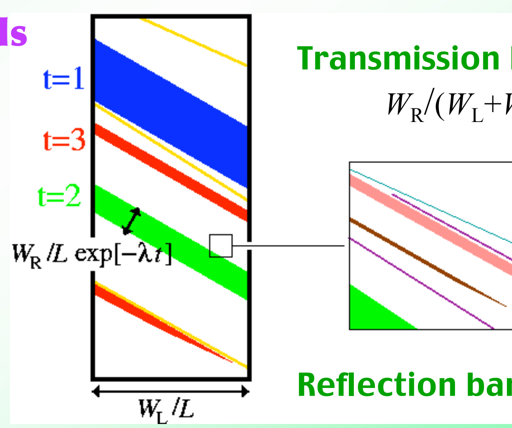

# Transmission bands in classical phase-space




**Liouville Blocking:**
If path goes $\boldsymbol{A \rightarrow B}$,
then no other path
can go to $\boldsymbol{B}$

**Transmission bands**

Area of band
$= W_L W_R / L^2 \exp[-\lambda t]$




Transmission bands fill
$W_R/(W_L+W_R)$ of left lead

Reflection bands fill the rest

## Basis states in phase-space

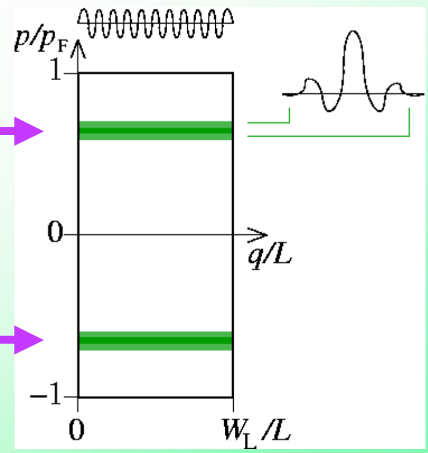
Plot a **basis state** in phase space as
its overlap with position/momentum states; $|\langle r|\psi_n\rangle\langle p|\psi_n\rangle|^2$

$\langle p|\psi_n\rangle$ is Fourier trans. of $\langle r|\psi_n\rangle$

➡ state's area in phase space is $\hbar$

Consider **basis** of leads modes;

$n$th mode of **left lead** is
**not localised** in phase-space

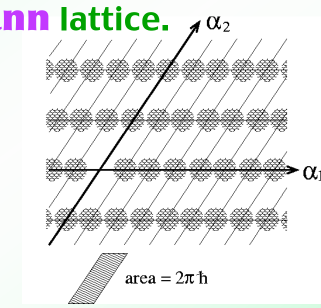



## Constructing a phase-space basis

Coherent state at each vertex of **von Neumann** lattice.

$<r|r_i;p_j> \propto \exp[\mathrm{i}\,\hbar^{-1} p_j\, r - (2\hbar)^{-1}(r - r_i)^2]$

Basis, $\{v_i\}$, is complete but not orthogonal
nearest neighbours have overlap = 0.21




**Now orthogonalise**
could use Gram–Schmit (ugly)

**Our Procedure :** (i) define **new** basis, $\{v'_i\}$, such that,

$$v'_i = A'_i\,[v_i - 1/2\ \textstyle\sum_{j\neq i} (v_i . v_j)\ v_j]$$

where $A'_i$ normalises $v'_i$ .

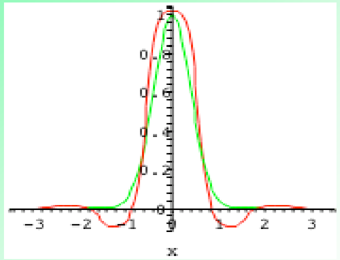
(ii) **repeat** procedure.

More usual approach : **wavelet analysis** (signal processing)

♣ Mathematician PROVED existence of such bases

♣ Engineer's use such bases – but they're also ugly

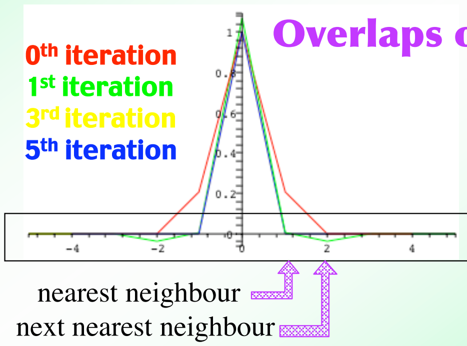
## Result of orthonormalisation



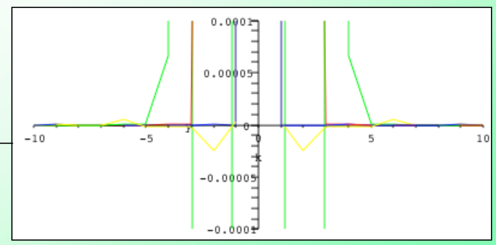
$$| \mathrm{ps}\ ; i,j > = \sum_{i'j'} \beta_{i'} \beta_{j'}\ |\mathrm{cs}; i'+i, j'+j >$$

$\beta_0 = \ \ 1.0357044$
$\beta_1 = -0.1130793$
$\beta_2 = \ \ 0.0174448$
$\beta_3 = -0.0030142$
$\beta_4 = \ \ 0.0005478$
$\beta_5 = -0.0001024$
$\beta_6 = \ \ 0.0000195$
$\beta_7 = -0.0000038$
$\beta_8 = \ \ 0.0000007$
$\beta_9 = -0.0000001$

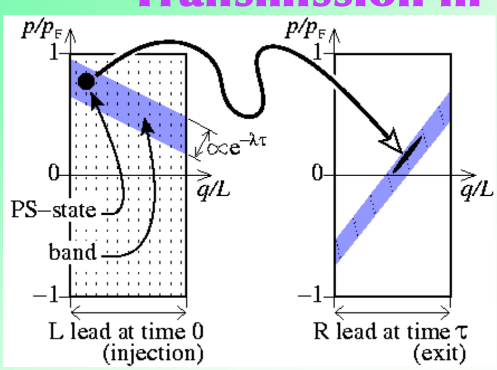
$|\beta_{n>9}| < 10^{-7}$




### Overlaps of states after each iteration



## Transmission in the phase-space basis

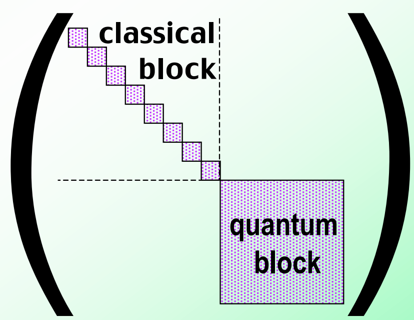



### Louville blocking

in classical phase space

➡ block diagonal **S**.

$$\mathbf{t' = V t U^\dagger} =$$



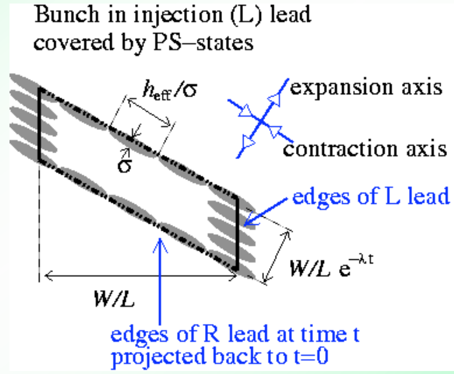

**classical modes** couple only to classical trajectories with $t < t_E^{op}$.
**quantum modes** couple "only" to trajectories with $t > t_E^{op}$.

➡ Transmission matrix, $\mathbf{T = t^\dagger t = U^\dagger (t'^\dagger t') U = U^\dagger T' U}$,

$\mathbf{T'}$ has same structure $\mathbf{t'}$ ➡ eigenvalues for classical modes

# Optimising the phase-space basis

**Optimally cover** each band:
Maximise # of ps-states in band dropping all states on boundary.




**Result :** orthonormal but **incomplete** basis

♣ **complete** inside bands with area $> \hbar$ ⇦ **classical modes**

♣ **doesn't cover** edge-of-bands states or multiple-bands states with areas $< \hbar$ ⇦ **quantum modes**

... then complete basis in some manner.

# Counting quantum and classical modes

Area of band, $\mathcal{A} = W_L W_R / L^2 \exp[-\lambda t]$ ;

Open cavity Ehrenfest time is time when $\mathcal{A} = \hbar_{\text{eff}}$

hence $t_E^{\text{op}} = \lambda^{-1} \ln\left[\hbar_{\text{eff}}^{-1} \times W_L W_R / L^2\right]$

**Three types of mode:**

(1) purely classical $N_{\text{cl}} \sim N\,(1-\exp[-t_E^{\text{op}}/\tau_D]) \sim \hbar^{-1}$

(2) purely quantum $N_{\text{qm}} \sim N \exp[-t_E^{\text{op}}/\tau_D] \sim \hbar^{-(1-1/\lambda\tau_D)}$

(3) on edge of classical region; $N'_{\text{qm}} \sim N_{\text{qm}}/\lambda\tau_D$

**In classical limit** (small $h$) ; $N_3 << N_2 << N_1$

As $h \to 0$ ; $N_1, N_2, N_3 \to \infty$ while $(N_2 + N_3)/N_1 \to 0$

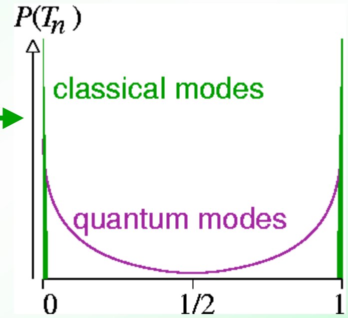
i.e. number of non-classical modes **diverges**,
but proportion of non-classical modes **vanishes**.

# Transmission spectrum and shot noise

**proven** classical modes have this form

**proven** quantum modes have
(i) separate subspace in scattering matrix
(ii) approx. quantum ergodicity
(spread over classical phase space)

**guess** quantum modes obey **RMT**.




**Shot Noise : noiseless** classical modes & **noisy** quantum modes

Shot noise, $S \sim N_{\text{quantum}} \sim N \exp[-t_E^{\text{op}}/\tau_D] \rightarrow \infty$

Fano factor, $F \sim N_{\text{quantum}}/N \sim \exp[-t_E^{\text{op}}/\tau_D] \rightarrow 0$

**OPEN QUESTION :** theory for **RMT** Transmission spectrum for quantum modes (all modes when $t_E$ is irrelevant)?

cf. weak localisation

# Intro. to weak localisation

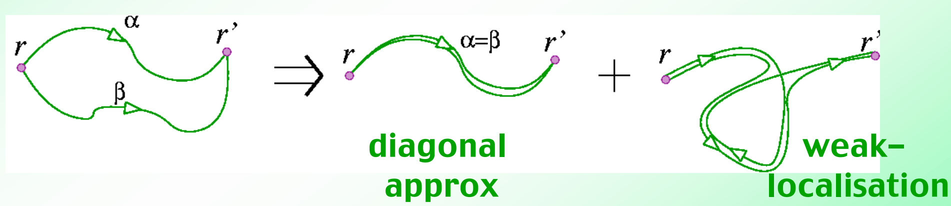
Larkin-Khmelnikski (1982), Richter-Sieber (2002)

Prob. to go from $r$ to $r'$ with energy $E$ in time $t$ is Fourier trans. of

$$G_R(r,r';E+\omega)G_A(r',r;E-\omega) = \sum_{\alpha\beta} A_\alpha A^*_\beta \exp[i(S_\alpha - S_\beta)/h]$$

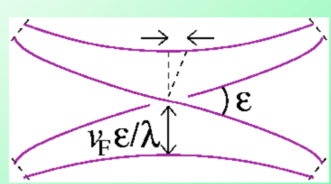
**Average** over $E$ or ensemble of macroscopically identical systems

➡ **cancellation** of all contributions with $S_\alpha - S_\beta >> h$




**Action difference:**

$$S_\alpha - S_\beta = 4 \times E_F\, \varepsilon^2/4\lambda$$



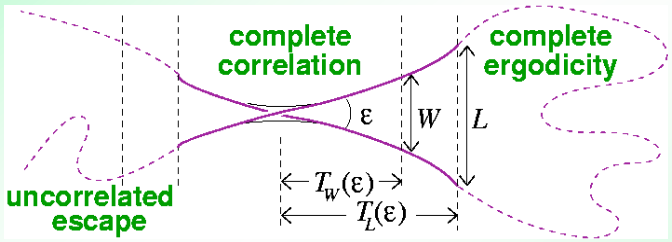
# Weak localisation with finite $t_E$

**two** cavity scales ; $W$ & $L$ ➡ **two** Ehrenfest times

$$t_E^{op} = \lambda^{-1} \ln\left[\hbar_{eff}^{-1} \times (W/L)^2\right] \quad \& \quad t_E^{cl} = \lambda^{-1} \ln\left[\hbar_{eff}^{-1}\right]$$

Prob. of crossing at times $t_1$ & $t_2$ at angle $\varepsilon$ is $(v_F^2 \sin \varepsilon / L^2)\, dt_1 dt_2\, d\varepsilon$

for $t_1 > T_W(\varepsilon)$ and $t_1 + 2T_L(\varepsilon) < t_2 < t - T_W(\varepsilon)$




Integral over $\varepsilon$ is dominated by

$$\varepsilon^2 \sim \hbar\lambda/E_F$$
$$T_W(\varepsilon) \sim t_E^{op}/2$$
$$T_L(\varepsilon) \sim t_E^{cl}/2$$

Prob. of escape at time $t$ is $\exp[-(t' - t_E^{op})/\tau_D]\ \Theta\,[-t' - t_E^{op}]$

where $t'$ is time during which escape is possible
i.e. not 2nd pass through complete correlation region.

# ... bringing it all together

weak localisation correction

$$\delta g \propto \frac{N_{qm}}{A_{qm}} \int_{t_E^{op}+t_E^{cl}}^{\infty} dt\ (t - t_E^{op} - t_E^{cl})^2 \exp[-(t - t_E^{op} - t_E^{op})/\tau_D]$$

Result is independent of **Ehrenfest time**

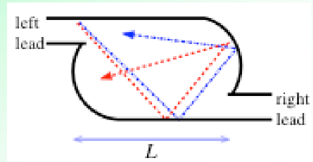
$$\delta g = -\frac{1}{4} \times [1 - \alpha(\lambda\tau_D)^{-1}]$$

RMT result

Non-RMT behaviour
Is $\alpha$ system specific or universal?

In good chaotic limit ( $\lambda\tau_D >> 1$ );
weak localisation correction is **independent of wavelength**
**does not decay** in deep classical limit (wavelength $\to 0$)

# Summary

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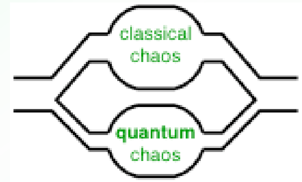


(i) **microscopic reason** for "two parallel cavities" behaviour

Require : Unitary rotation to **phase-space basis**

& **Liouville blocking** in classical phase-space

(ii) Number of modes in each cavity

$$N_{\mathrm{qm}} = N \exp[-t_E^{\mathrm{op}}/\tau_D] \quad \& \quad N_{\mathrm{cl}} = N - N_{\mathrm{qm}}$$

In classical limit; $N_{\mathrm{qm}} \to \infty$ but $N_{\mathrm{qm}}/N_{\mathrm{cl}} \to 0$

(iii) Fano factor (shot–noise) $= 1/4 \times \exp[-t_E^{\mathrm{op}}/\tau_D]$

weak localisation $= -1/4$

no factor of $\exp[-t_E^{\mathrm{op}}/\tau_D]$