Berry phase in the presence of external noise

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Abstract. Here we summarise and place in context two of our earlier works. There we investigate the geometric phase or Berry phase (BP) acquired by a spin-half which is subject to external noise in addition to a slowly varying magnetic field (which generates the Berry phase). The noise may be due to the fluctuations of either quantum or classical degrees-of-freedom in the vicinity of the spin. We find that the noise *modifies* the Berry phase and that this modification is of a geometric nature. While the original BP (for an isolated system) is the flux of a *monopole*-field through the loop traversed by the magnetic field, the noise-induced modification of the BP is the flux of a *quadrupole-like* field. The noise-induced phase is *complex* and its imaginary part giving a geometric contribution to dephasing; its sign depends on the direction of the loop. Unlike the BP, this *geometric dephasing* is gauge invariant for open paths of the magnetic field. We discuss the consequences of this for proposals to use Berry phases to control qubits, in particular solid state (superconducting) qubits.

Keywords: geometric phases, dissipative quantum mechanics, decoherence, qubits
PACS: 03.65.Vf, 03.65.Yz, 85.25.Cp

Introduction: Rather than present new results in this conference proceedings, we take the opportunity to summarise and place in context two of our earlier works [1, 2].

There has recently been significant progress in building quantum devices over which one can have complete *phase-coherence* control [3]; much of the interest in such devices is in their potential as qubits [4]. This work has renewed interest in the effect of external noise on quantum systems. It has been known for a long time that such noise — induced by the coupling of the system to environmental degrees-of-freedom beyond experimental control — leads to non-Hermitian evolution of the state of the system [5, 6, 7]. For a two-level system, weak external noise causes energy-relaxation on a timescale T_1, loss of coherence on a timescale T_2, and a Lamb-like shift of the energy-levels, $\delta B_{\rm Lamb}$. Even relatively weak external noise can induce a T_2 which is too short to allow coherent control of the quantum system. The challenge is to understand how the noise affects quantum systems, with a view to finding methods for minimising the loss of coherence.

Here we study the effect of external noise on the Berry phase [8, 9]. Berry showed that the phase acquired by an eigenstate of a slowly varying Hamiltonian $H(t)$ is related to the geometric properties of the loop traversed by $H(t)$. It was suggested [10] that the *Berry phase would be insensitive to external noise*. The argument was that the noise-induced fluctuations of geometric properties of the loop traversed by $H(t)$ would average away on the long times required for an adiabatic Berry phase experiment. It had been shown

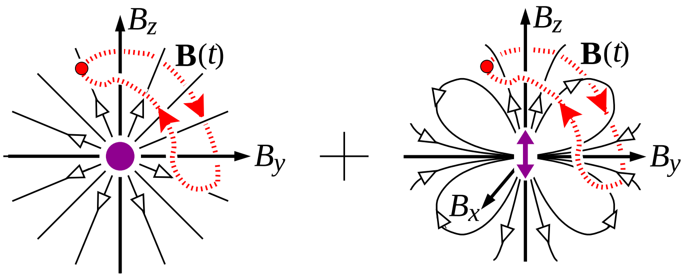

FIGURE 1. Without external noise, the Berry phase is the flux enclosed by the path of $\mathbf{B}(t)$ (shown as the discontinuous line) due to a monopole field (the continuous lines in the left-hand plot). For a system experiencing anisotropic external noise, the Berry phase is the flux enclosed by the path of $\mathbf{B}(t)$ due to the sum of the monopole field *and a quadrupole-like field* (the latter is shown on the right). The quadrupole-like field is the *noise-induced* Berry phase; its magnitude goes like the strength of the coupling to the environment squared. The prefactor on this term is *complex*, leading to geometric dephasing.

that Berry phases could be used for complete control of the state of a two-level system (qubit) [11, 12]. Thus it seemed that even in the presence of external noise coherent control of a qubit could be achieved using Berry phases.

Independently we approached the problem from exactly the opposite direction. It is well known that to manipulate a quantum system adiabatically one must change the Hamiltonian at a rate much less than the spectral gap. However a quantum system in a noisy environment has broadened levels and hence no true gap. Thus one would naively assume that the condition for adiabaticity can never be satisfied, and hence even *infinitesimal external noise would destroy the Berry phase*. This is clearly too naive, all real systems experience some external noise and yet Berry phases have been observed.

The contradiction between these two naive arguments led us to investigate the exact conditions for observing a Berry phase in a system which experiences external noise [1], these are summarised below. We then proceeded to investigate the nature of the Berry phase, and have shown that it is *modified by the external noise* [1], that this modification is *geometric and complex*[2].

Berry phase without noise: Berry [8] considers a two-level spin-half system in a magnetic field, which is varied slowly along a closed path: $H_{\rm spin} = -\frac{1}{2}\mathbf{B}(t)\sigma$ [13]. The rate of the field's change is characterized by the time to complete the loop, $t_{\rm P}$. In the adiabatic limit, $Bt_{\rm P} \gg 1$, the relative phase acquired by the eigenstates is a sum $\Phi = \oint |\mathbf{B}(t)|dt + \Phi_{\rm BP}$ of the dynamical and Berry phases. The latter is geometric, it depends on the geometry of the loop but not on the details of its traversal (for an isolated spin-half it is given by the solid angle subtended by the loop $\mathbf{B}(t)$). In the spin language, the evolution is a rotation of the spin by an angle Φ about $\mathbf{B}$. Note there are non-adiabatic correction to the phase (and to the amplitude to return to spin-up at $t_{\rm P}$) which go like $(Bt_{\rm P})^{-1}$, these can only be ignored in the adiabatic limit, $Bt_{\rm P} \gg 1$.

Observing the Berry phase in the presence of external noise: We consider a spin-half coupled to a time-dependent field that we slowly rotate along a closed loop, $\mathbf{B}(t)$ and a noisy-field along the z-axis, $\hat{\mathbf{X}}(t) \equiv \hat{X}(t)\mathbf{z}$. The Hamiltonian reads [13]

$$\hat{H} = -\frac{1}{2}\mathbf{B}(t)\hat{\sigma} - \frac{1}{2}\hat{X}\hat{\sigma}_z + \hat{H}_{\rm env}(\hat{X})\ . \qquad (1)$$

Below we discuss the generalisation of this to multiple noisy-fields coupled to multiple axes, for now we consider a single noise-field as in the above Hamiltonian.

We showed in [1] that the condition for observing the Berry phase of a system is that the external noise is weak enough that $B \gg T_2^{-1}$. Then one can carry out a BP experiment slowly enough for non-adiabatic effects to be ignored ($t_P \gg B^{-1}$) while fast enough that the noise has not completely destroyed the coherence, i.e. it has not destroyed the phase information ($t_P \lesssim T_2$). If the noise is stronger than this there is no t_P for which one can observe the BP of the quantum system.

Why anisotropic noise is highly-relevant to experiments: In our works we consider anisotropic noise; indeed in [2] we noted that for isotropic external noise the modification of the BP is *zero*, see below. Thus one should ask whether anisotropic noise is relevant to experiments. However we argue that situations in which the noise is *isotropic* will be extremely rare, in most real solid-state devices the noise is strongly anisotropic. Take for example the super-conducting charge qubit of Nakamura *et al* [3] (often called a Cooper-pair box); it is an effective two-level system which has either n or $(n+1)$ Cooper pairs on a superconducting island which forms one arm of a SQUID. One can treat it as a pseudo-spin-half with the z-axis effective field given by a gate voltage and the x-axis effective field given by a flux through the SQUID. It is then clear that there is no reason that the magnitude of the noise in the x- and z-directions should be related to each other; the former being caused by charge fluctuations and the latter being caused by noise in the applied magnetic-field. More generally, the strongest source of noise is often fluctuations of the field used to control the quantum system, since these fields are not isotropic, it seem unlikely that the noise in them will be.

Difference between external noise of quantum and classical origin: We define two noise correlators $S_\pm(t;\tau) = \langle [\hat{X}(t+\tau)\hat{X}(t) \pm \hat{X}(t+\tau)\hat{X}(t)] \rangle$, where for noise of classical origin $\langle \cdots \rangle$ indicates averaging, while for noise of quantum origin $\langle \cdots \rangle$ indicates an expectation value. For noise generated by an environment containing many non-identical degrees-of-freedom, one often finds that these correlators have a finite correlation time; i.e. they decays on a timescale $t_{\rm mem}$, often called the memory time. If in such cases $\langle \hat{X}^2 \rangle t_{\rm mem} \ll B$ then the effect of the noise is weak enough that it is given by golden-rule type calculations in which all effect of the noise on the system's dynamics are given by these two noise correlators [14]. All other details of the environment (oscillators/two-level-systems/chaotic-system, fermionic/bosonic, etc) are irrelevant.

The only difference between noise of classical origin and noise of quantum origin, is that in the former case $S_-(t;\tau) = 0$ while in the latter case this may be non-zero. We showed in [2] (see also [15]) that the noise-induced modification of the Berry phase is independent of $S_-(t;\tau)$. Thus the Berry phase is insensitive to whether the noise is of quantum or classical origin, it depends only on the functional form of $S_+(t;\tau)$.

Modification of Berry phase: In the absence of noise, the BP [8] is given by the surface integral $\Phi_{\rm BP} = \int {\rm d}\mathbf{S}\, \mathbf{b}(\mathbf{B})$, where the field $\mathbf{b}(\mathbf{B})$ is that of a charge-one monopole, $b_B = 1/B^2$. We find that the noise induces an additional BP of the form $\delta\Phi_{\rm BP} = \int {\rm d}\mathbf{S}\, \delta\mathbf{b}(\mathbf{B})$ where the field $\delta\mathbf{b}(\mathbf{B})$ has two non-zero components;

$$\delta b_B = B^{-2} F(B) (3\cos^2\theta - 1) \qquad \delta b_\theta = -B^{-1} F'(B) \sin\theta\cos\theta \tag{2}$$

The function $F(B)$ is a rather ugly function of the Fourier transform of $S_+(\tau)$, see [2].

The angular dependence of $\delta\mathbf{b}$ resembles that of a *quadrupole*. For slow environment modes ($\Omega \ll B$), $\mathrm{Re}[F(B)] \propto B^{-2}$ and hence $\delta\mathbf{b}(\mathbf{B})$ is a quadrupole field. For other environment modes, $\delta\mathbf{b}$ has non-zero curl, thus it is not a sum multipoles but rather the field generated by a pseudo-current [2].

Here we briefly summarise the steps of the derivation of the above result, and refer those interested in details to [2]. We go to a rotating frame, as used in [16], which is chosen in such a way that $\mathbf{B}(t)$ remains static in the new frame. The price we pay is that this new frame is non-inertial, rotating with angular velocity vector, ω; so in this frame the spin experiences an additional pseudo-magnetic-field, ω. However since we are in the adiabatic limit of slow rotation, $\omega \ll B$ and so the total (noiseless) field in this frame, $(\mathbf{B}+\omega)$ is approximately time-independent. Thus we can use the usual techniques for introducing external noise into an otherwise static Hamiltonian [5], which lead to a dissipative master equation for the spin-half. The time-dependence of ω and the axis that X couples to only slightly complicate the calculation. The total (complex) phase acquired by the off-diagonal elements of the density matrix is given by the Lamb-shifted (complex) energy gap (in the rotating frame) integrated over time. The imaginary part of this energy is T_2^{-1} and gives the width of the levels. However both the real part of the noise-induced shift, δB_{Lamb}, and the imaginary part, T_2^{-1}, are a function of the initial gap, which in the rotating frame is $|\mathbf{B}+\omega|$; thus expanding $(\delta B_{\mathrm{Lamb}}+\mathrm{i}T_2^{-1})$ to first order in ω leads to an order $X^2\omega$-term which when integrated over time gives a *complex* geometric phase [17]. Thus one can write the total (complex) BP for a closed path as

$$\Phi_{\mathrm{BP}} = \oint \mathrm{d}\varphi \, \frac{\mathrm{d}}{\mathrm{d}B_z}\left(B+\delta B_{\mathrm{Lamb}}+\mathrm{i}T_2^{-1}\right) . \tag{3}$$

where B_z is the component of $\mathbf{B}$ along the axis that the noise couples to (the z-axis).

Geometric dephasing: The imaginary part of a phase causes the decay of off-diagonal elements of the density matrix, and hence is dephasing. In eq. (3) we clearly see that the noise introduces an imaginary part to the BP, this imaginary geometric term is *geometric dephasing*.

The first intriguing feature of this geometric dephasing is that it may be of *either sign* (it changes sign if $\mathbf{B}(t)$ traverses the same loop in the opposite direction). If this were the only source of dephasing, it would be unphysical because it could lead to the purity of the density matrix becoming greater than one. However it is not the only source of dephasing; it is a small modification of the total dephasing. Our analysis is in the adiabatic limit $Bt_{\mathrm{P}} \gg 1$, and the total dephasing goes like $\langle\hat{X}^2\rangle Bt_{\mathrm{P}}$ while the geometric dephasing only goes like $\langle\hat{X}^2\rangle$. None-the-less this asymmetry in the dephasing (between going one way or the other way around a loop) is unexpected and intriguing.

The second intriguing feature of the geometric dephasing is that it is gauge-*independent* for *open* paths of $\mathbf{B}(t)$. This is absolutely different from the real part of the BP, its gauge-*dependence* is a consequence of ambiguity in the choice of instantaneous basis of cartesian coordinates for a given $\mathbf{B}(t)$; $\mathbf{B}(t)$ defines the instantaneous z-axis but the x-axis may lie anywhere in the plane perpendicular to $\mathbf{B}(t)$. Gauge transformations rotate this x-axis in this plane. The ambiguity is absent for closed loops, where the final basis must coincide with the initial one. However one can monitor the magnitude of the transverse spin component (dephasing); this magnitude is independent of the choice of the instantaneous x-axis, thus *all dephasing is gauge-invariant for open paths*.

Note that the BP above, in eqs. (2,3), is *only* for *closed* paths of $\mathbf{B}(t)$. This is because it was derived using Stokes' theorem to write the line-integral along a *closed* path as a surface integral. The result for $\mathrm{Im}[\Phi_{\mathrm{BP}}]$ for *open* paths is given in [2] and is rather uglier than those above.

Multiple sources of external noise: Above we considered a single noise field coupled to the z-component of the spin. However since $\mathbf{B}(t)$ was left arbitrary the analysis is equally applicable if the noise field couples to any axis, i.e. to $\sigma_r \equiv \mathbf{r}.\sigma$ where $\mathbf{r}$ could be any unit vector. In this case the axis of the quadrupole-like field would simply be along the $\mathbf{r}$-axis instead of the z-axis.

It is then straightforward to generalise our analysis to multiple noisy-fields coupling to multiple arbitrary components of the spin. In this case we see that each noise field's effect on the BP is independent (at least to the order in coupling that we do the calculations) and so their effects on the BP just sum up. Thus the resulting noise-induced BP is the sum of the quadrupole-like fields each with a different magnitude and each with their axis pointing in a different direction.

There is a interesting special case of this, and that is an *isotropic* environment. This is modelled by an environment coupling of the form $\mathbf{X}.\sigma$ where $\mathbf{X}$ has three components and the environment Hamiltonian is of the form $\hat{H}_{\mathrm{env}}(|\mathbf{X}|)$. It is clear to see that this Hamiltonian has no preferred direction in space, so there can be no quadrupole-like field. Indeed the only field which can have no preferred direction in space is the monopole-field. Thus one might expect the magnitude of the monopole field to be modified by the presence of the environment. However the BP must be insensitive to whether one defines it via the interior or exterior solid angle (up to factors of 2π); hence the monopole field must have a magnitude of one and so it cannot be modified by the environment. A more pedestrian way to see this is to assume the isotropic environment has a Hamiltonian of the form $\hat{H}_{\mathrm{env}}(X_x)+\hat{H}_{\mathrm{env}}(X_y)+\hat{H}_{\mathrm{env}}(X_z)$. Then the x-component of the environment generates a quadrupole-like field with its axis in the x-direction, the y-component generates one with its axis in the y-direction, etc. The three quadrupoles all have the same magnitude, and they cancel each other when we sum them up.

Acknowledgments: This work is part of the CFN (DFG) and was supported by the EC RTN Spintronics, and by the ISF of the IAS. RW was supported by the EPSRC and the NSF. YM was supported by the AvH Foundation (S. Kovalevskaya award). and YG is supported by the AvH Foundation (Max-Planck Award). RW, YM, AS acknowledge partial support from the Minerva Einstein center (DFG) during their visits to the Weizmann Institute, while YM and AS also thank the Transnational Access program for the support during a visit to the WIS.

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