Chaotic transport: from quantum to classical

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Abstract. We present a semiclassical theory for the scattering matrix $\mathscr{S}$ of a chaotic ballistic cavity at finite Ehrenfest time. Using a phase-space representation we show that the Liouville conservation of phase-space volume decomposes $\mathscr{S}$ as $\mathscr{S} = \mathscr{S}^{\rm cl} \oplus \mathscr{S}^{\rm qm}$. The short-time, classical contribution $\mathscr{S}^{\rm cl}$ generates deterministic transmission eigenvalues $T = 0$ or 1, while quantum ergodicity is recovered within the subspace corresponding to the long-time, stochastic contribution $\mathscr{S}^{\rm qm}$. This provides a microscopic foundation for the two-phase fluid model, in which the cavity acts like a classical and a quantum cavity in parallel. Our model shows that the Fano factor of the shot-noise power vanishes in this limit, while weak-localization remains universal.

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Introduction: In recent years it has been possible to make electronic systems clean enough that the electron have a mean free path significantly longer than the size of the potential that confines them[1]. The electrons then move ballistically in these *quantum dots*, in a manner strongly related to *classical* dynamics in the dot. When this classical motion is *chaotic* the transport properties are usually universal and well-captured by random matrix theory (RMT). However once the *Ehrenfest time* becomes a relevant parameter this universality is broken and the transport properties cease to be described by RMT [2]. Elsewhere [3, 4] we explore the role played by the Ehrenfest time in a clean chaotic system connected to two leads in the limit where the Fermi wavelength is much smaller than all system lengthscales (system size, lead widths). Here we define open-cavity Ehrenfest times in such systems and emphasise their relevance to the transport properties. We then show that for finite open-cavity Ehrenfest time the cavity scattering matrix is block diagonal and hence behaves like two cavities in parallel. One of these cavities is classical in nature the other is quantum. We prove that the classical cavity's transmission eigenvalues are all zero or one, so its transport properties are *deterministic* and hence *noiseless*. Meanwhile the quantum cavity is *stochastic*, with its transport properties exhibiting *quantum noise*. Finally we touch on the consequences of this for the Fano factor and the quantum (weak localisation) correction to conductance.

Ehrenfest times : Ehrenfest times are the time-scales on which quantum effects start to become relevant in the evolution of a wavepacket. We consider a chaotic cavity of size L and Lyapunov exponent λ which connected to two leads (Left and Right) with widths $W_{\rm L}, W_{\rm R}$; where $L, W_{\rm L}, W_{\rm R}$ are all much larger than the Fermi wavelength, $\hbar/p_{\rm F}$. There are two *open-cavity* Ehrenfest times [5] associated with modes entering the cavity from the Left (L) lead;

$$\tau_{\rm E}^{\rm LR} = \lambda^{-1} \ln[\hbar_{\rm eff}^{-1}(W_{\rm L}W_{\rm R}/L^2)], \qquad \tau_{\rm E}^{\rm LL} = \lambda^{-1} \ln[\hbar_{\rm eff}^{-1}(W_{\rm L}^2/L^2)], \tag{1}$$

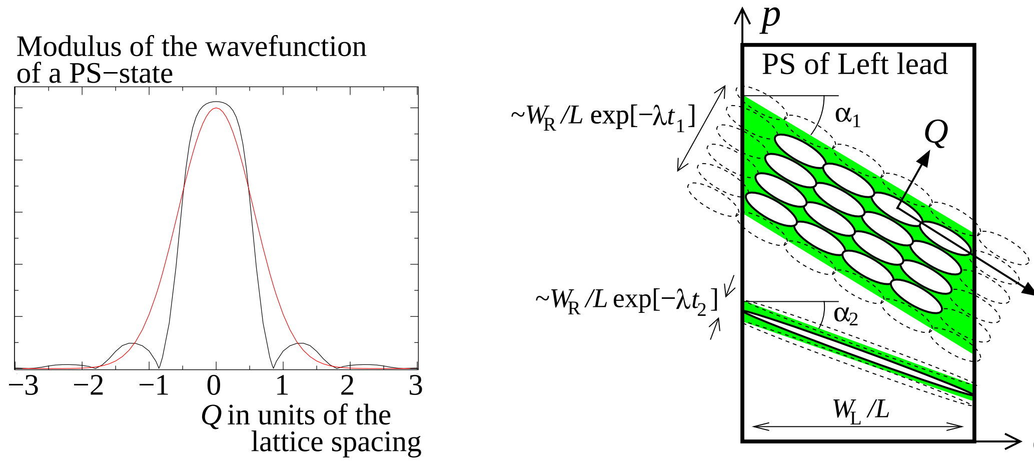

FIGURE 1. **On the left** is a plot of a PS-state as a function of dimensionless position (dark line); for comparison we plot the wavefunction of the coherent state (grey line). Both wavefunctions have the same shape as a function of P as they do as a function of Q (up to a scaling factor). The PS-state's oscillations make it orthogonal to PS-states centred at finite Q, and its broadened peak (w.r.t. the coherent state) makes it orthogonal to PS-states centred at finite P. **On the right** we show two bands on the Left lead (in grey), with PS-states super-imposed on them (ellipses). The lattice of PS-states has been stretched/rotated to maximise the number of PS-states in each band (solid-edged ellipses) while minimising the number partially in each band (dashed-edged ellipses). Thus the PS-states have the same aspect ratio as the band.

where the dimensionless Planck constant $\hbar_{\rm eff} = \hbar/(p_{\rm F}L)$. The first time is for *transmission* (L to R) and the second is for *reflection* (L to L). In addition there is the *closed-cavity* Ehrenfest time [2,6-9], $\tau_{\rm E}^{\rm cl} = \lambda^{-1}\ln[\hbar_{\rm eff}^{-1}]$, unlike those above it is a property of the cavity itself and is independent of the size of the leads.

The three time-scales can be derived as follows. We assume the cavity is a two-dimensional hyperbolic chaotic system. Then the Poincaré surface of section perpendicular to any trajectory is a two-dimensional phase space $(r_\perp, p_\perp)$, which we make dimensionless by writing distances in units of L and momenta in units of $p_{\rm F}$. Then the Liouvillian flow on the Poincaré surface of section stretches exponentially, with rate λ in the *unstable* direction, while compressing exponentially in the *stable* direction. The Ehrenfest times are then given by $\lambda^{-1}\ln[\hbar_{\rm eff}^{-1}XY]$ where X and Y are dimensionless system lengthscales; $W_{\rm L}/L$, $W_{\rm R}/L$ or 1. This is the time for a wavepacket with width X in the *stable* direction (and hence $\hbar_{\rm eff}/X$ in the *unstable* direction) to spread under the Liouvillian flow to width Y in the *unstable* direction. We note that for all times of relevance here, the evolution of wavepackets inside the system is well approximated by the Liouvillian flow of the *classical* dynamics.

Bands in the classical phase-space (PS): The finiteness of $\tau_{\rm D}$ (the dwell time for trajectories in the cavity) means that classical trajectories injected into a cavity are naturally grouped into PS transmission and reflection bands [12], despite the ergodicity of the associated closed cavity. Each band on the PS cross-section of the L lead (see Fig. 1) consists of a group of classical paths which exit through the same lead after the same number of bounces, τ, (having followed similar paths through the cavity). Because of the chaotic classical dynamics, bands with longer escape times are narrower, having a width (and hence a PS area) scaling like $\propto \exp[-\lambda\tau]$. The Ehrenfest time is the time at which this area becomes smaller than $\hbar_{\rm eff}$. Thus only for times *shorter* than this can a band carry one (or more) *whole* quantum wavepacket.

Counting classical and quantum modes in the scattering matrix: All trajectories which exit through the R lead at $\tau < \tau_{\rm E}^{\rm LR}$ will be in bands with phase-space area larger than $\hbar_{\rm eff}$. We show below that these modes are *classical*. Thus the number of transmitting (reflecting) *classical* PS-states is given by the area of the L lead's phase-space which couples to transmitting (reflecting) trajectories with $\tau < \tau_{\rm E}^{\rm LR}$ (with $\tau < \tau_{\rm E}^{\rm LL}$). The total number of classical modes in the L lead is the sum of these two;

$$N_{\rm L}^{\rm cl} \;=\; [N_{\rm L}+N_{\rm R}]^{-1}\left[N_{\rm L}^2(1-{\rm e}^{-\tau_{\rm E}^{\rm LL}/\tau_{\rm D}})+N_{\rm L}N_{\rm R}(1-{\rm e}^{-\tau_{\rm E}^{\rm LR}/\tau_{\rm D}})\right] \tag{2}$$

All other modes of the L lead sit over many transmission bands with $\tau > \tau_{\rm E}^{\rm LR}$ or reflection bands with $\tau > \tau_{\rm E}^{\rm LL}$, and so they are *quantum* PS-states; thus $N_{\rm L}^{\rm qm} = N_{\rm L} - N_{\rm L}^{\rm cl}$. We can do the same for the phase-space of the R lead by swapping L and R throughout.

The phase-space basis: Below we write the scattering matrix $\mathscr{S}$ in a PS-basis, whose construction we now summarize, for details see [4]. We construct the PS-basis by covering all phase-space bands with area bigger than $\hbar_{\rm eff}$ with a lattice of PS-states of the form shown in Fig. 1. The lattice is stretched and rotated to optimally cover each band (as in Fig. 1). All states on the lattice covering each such band are *orthonormal*, and the basis is *complete* on the parts of phase space which are covered by these bands. The position of the lattice on each band is chosen such that each ingoing PS-state evolves under the cavity dynamics to exit as exactly one outgoing PS-state. Each basis states exits at a time less that $\tau_{\rm E}^{\rm LR}$ (for transmission) or $\tau_{\rm E}^{\rm LL}$ (for reflection). It exit as a single wavepacket at a single time; thus it behave *deterministically*; i.e. like a classical particle with its quantum nature completely hidden.

The remaining phase-space (made of classical bands with phase-space area less than $\hbar_{\rm eff}$) is covered by states chosen simply to complete the basis. The fact that the basis is already complete on the bands with area larger than $\hbar_{\rm eff}$, means that each remaining PS-states must sit on many bands in the classical phase space which exit at many different times. Thus these PS-basis states exhibit strongly quantum behaviour.

Scattering matrix in the PS-basis: The transformation from the basis of lead modes to the PS-basis is *unitary* because both bases are complete and orthonormal. Thus this transformation leaves *unchanged* the eigenvalues of the scattering matrix, $\mathscr{S}$, and the transmission matrix $\mathscr{T} = \mathbf{t}^\dagger\mathbf{t}$ (where $\mathbf{t}$ is the L to R transmission block of $\mathscr{S}$). In the PS-basis, the scattering matrix takes the form

$$\mathscr{S} = \mathscr{S}_{\rm cl} \oplus \mathscr{S}_{\rm qm} = \begin{pmatrix} \mathscr{S}_{\rm cl} & 0 \\ 0 & \mathscr{S}_{\rm qm} \end{pmatrix} \tag{3}$$

The matrix $\mathscr{S}_{\rm cl}$ is $N^{\rm cl}\times N^{\rm cl}$ while the matrix $\mathscr{S}_{\rm qm}$ is $N^{\rm qm}\times N^{\rm qm}$, with $N^{\rm cl} = N_{\rm L}^{\rm cl} + N_{\rm R}^{\rm cl}$ and $N^{\rm qm} = N_{\rm L}^{\rm qm} + N_{\rm R}^{\rm qm}$. The matrix $\mathscr{S}_{\rm cl}$ must have only one non-zero element in each row and column. After re-ordering the labels of the modes on L and R, we can write

$$\mathscr{S}_{\rm cl} \;\equiv\; \begin{pmatrix} \mathbf{r}_{\rm cl} & \mathbf{t}'_{\rm cl} \\ \mathbf{t}_{\rm cl} & \mathbf{r}'_{\rm cl} \end{pmatrix} \qquad \mathbf{t}_{\rm cl} = \begin{pmatrix} \tilde{\mathbf{t}}_{\rm cl} & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{r}_{\rm cl} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\mathbf{r}}_{\rm cl} \end{pmatrix} \tag{4}$$

The matrices $\tilde{\mathbf{t}}_{\rm cl}$ and $\tilde{\mathbf{t}}'_{\rm cl}$ are $n \times n$, where $n = [N_{\rm L}N_{\rm R}/(N_{\rm L}+N_{\rm R})]\exp[-\tau_{\rm E}^{\rm LR}/\tau_{\rm D}]$ is the number of *classical transmission modes*. The matrix $\tilde{\mathbf{r}}_{\rm cl}$ is $(N_L^{\rm cl}-n)\times(N_L^{\rm cl}-n)$ and $\tilde{\mathbf{r}}'_{\rm cl}$ is $(N_R^{\rm cl}-n)\times(N_R^{\rm cl}-n)$. The matrix $\tilde{\mathbf{t}}_{\rm cl}$ is diagonal with elements given by $\tilde{t}_{ij} = {\rm e}^{{\rm i}\Phi_i}\delta_{ij}$ The matrix $\tilde{\mathbf{r}}_{\rm cl}$ has a more complicated structure; it has *exactly one* non-zero element in each row and each column. Thus we have diagonalised $N_{\rm L}^{\rm cl}$ of the modes of the transmission matrix, $\mathscr{T}$. It has n modes with eigenvalue $T_\alpha = 1$ and $N_{\rm L}^{\rm cl}-n$ modes with eigenvalue $T_\alpha = 1$. As noise $\propto \sum_\alpha T_\alpha(1-T_\alpha)$, all these modes are noiseless. In the classical limit the proportion of such classical (noiseless) modes goes to one [10]. The remaining modes (which remain numerous despite their proportion going to zero) are quantum in nature and are unitary *within* their own subspace, $\mathscr{S}_{\rm qm}$ [11].

Average conductance: All *transmitting* quantum and classical modes carry current, so the dimensionless conductance equals $N_{\rm L}N_{\rm R}/(N_{\rm L}+N_{\rm R}) \propto \hbar^{-1}$.

Zero-frequency noise and the Fano factor : As the classical modes are *noiseless*, all noise is generated by the quantum modes. The number of quantum (noisy) transmission modes is $[N_{\rm L}N_{\rm R}/(N_{\rm L}+N_{\rm R})]\exp[-\tau_{\rm E}^{\rm LR}/\tau_{\rm D}]$ goes to infinity in the classical limit $\hbar \to 0$. However the Fano factor $\propto$ (noise/average current) scales like $\exp[-\tau_{\rm E}^{\rm LR}/\tau_{\rm D}]$, vanishing as $\hbar \to 0$. This fits numerical and experimental [13] observations and has qualitative agreement with the earlier microscopic theory [6].

Weak localisation (WL): Recent numerics [14] have called into question previous microscopic theories [2, 9] which predicted that the WL correction to conductance decays like $\exp[-\tau_{\rm E}^{\rm cl}/\tau_{\rm D}]$. We perform a calculation similar to [8, 9] for the quantum modes (the classical modes have no WL correction); for details see [4]. The result in eq. (3), provides a factor of $\tau_{\rm E}^{\rm LR}$ which cancels the $\tau_{\rm E}^{\rm cl}$ in the exponent, leading to the universal (RMT) result even as $\hbar \to 0$ (for well developed chaos, $\lambda\tau_{\rm D} \gg 1$).

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