Project : **ERRORS AND FIDELITY IN QUBITS**

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During the project you should do most the things on the list. The points marked with a ♠ are more difficult, you will probably need to talk to me about them.

1 What is a qubit? Why are errors in qubits important?

To make an ordinary computer one needs to connect together many bits in such a way that one can perform operations on them. Each bit is in the state zero or one. To make a *quantum computer*, one needs to connect together many "qubits" (quantum bits). Each qubit can be in any *superposition* of zero and one ($|0\rangle$ and $|1\rangle$), i.e. it can be in a state $|\sigma\rangle = u\,|0\rangle + v\,|1\rangle$ where u and v are complex numbers. Thus the qubit is a spin-half where we choose $|\uparrow\rangle$ to be "zero" and $|\downarrow\rangle$ to be "one". Thus we write

$$|\uparrow\rangle = |0\rangle \quad \text{and} \quad |\downarrow\rangle = |1\rangle \tag{1}$$

In an ordinary computer one performs operators on the bits. The NOT gate is a single bit operation (it makes $0 \to 1$ and $1 \to 0$), while the XOR gate is a two bit operation (the output depends on the input state of two bits). In a quantum computer one also has one- and two-bit operations. There are two crucial one-bit operations; the *Hadamard gate* and the *phase gate*. There are various two-bit operations such as *controlled-phase shift gate*, etc, but I will not discuss them here. One-bit operations can be carried out by applying magnetic fields to the spin-half. These fields will cause the spin to precess, if we apply the field for a fixed length of time we can change the state of the spin however we like.

A crucial difference between an ordinary computer and a quantum computer is how sensitive they are to errors. The quantum computer is much more sensitive to errors. This makes building a quantum computer an extremely difficult task. Some very smart guys have invented error correction codes, which will fix any small errors in qubits. However for them to work the errors must be less than 10^{-4}, i.e. every qubit operation must be controlled to an accuracy better that 0.01%. The challenge facing physicists is how to make qubits which can be controlled with this accuracy.

In this project I would like you to think about the errors which are caused by slight imperfections in the way one manipulates each qubit. For example:

- Systematic errors: If the evolution is in a slightly incorrect field (maybe my apparatus is gives a field which is 0.1% wrong), how big will the error be in the qubit?
- Random errors: Suppose the fields are different each time one uses the qubit. What will the average error be in the qubit, what will be the variance of the error?

To quantify the errors in the qubit we use the concept of *FIDELITY*. It is a measure of how close the real output of a device is to the desired output[1]. For a quantum system, the fidelity is given by the *overlap* of the real outcome of the qubit operation and the desired outcome. If the desired outcome is a state $u\,|\uparrow\rangle + v\,|\downarrow\rangle$, but the real outcome is $u'\,|\uparrow\rangle + v'\,|\downarrow\rangle$ because of errors in the qubit, then the fidelity $F = \left|(u^*, v^*) \cdot \begin{pmatrix} u' \\ v' \end{pmatrix}\right|^2$. If $F = 1$ the qubit is perfect; small F equals big errors.

[1] The fidelity of your stereo's loud-speakers is a measure of how close the output sound is to the "perfect" output. There the "perfect" output sound is an exact copy of input signal.

2 BACKGROUND READING

♢ Search the **internet** for websites about "quantum computers".
(Do not spend more than 2 hours doing this).
Good keywords: quantum computing, quantum computer, quantum bit, qubit or q-bit.

♢ Search on **web of science** for introductory articles on "quantum computers".
address : *http://wos.consortium.ch/* (Do not spend more than 2 hours doing this).

Good keywords: (quantum computing/computer, qubit, etc) + (introduction, review)

Putting "introduction" or "review" will remove the most of the extremely technical articles.

♢ Read what you found, but **more importantly read [1-2]**. Understand the *Hadamard gate* and the *phase gate*[**2**].

♠ Understand the *controlled not gate* and *controlled phase-shift gate* [**2**].

Understand that error correction exists (but you DO NOT need to understand how it works).

DO NOT try to understand quantum computing algorithms such as Shor algorithm.

♠ Get an idea of how qubits are constructed in reality; using superconducting nano-circuits (squids) [**3**] or electrons on quantum dots [**4**]. Details are not important, just get a feel for how much physics is involved.

[**1**] Introduction of "Quantum Computing" A. M. Steane, Reports on Progress in Physics, vol 61, pp 117-173 (1998) (online at *http://arxiv.org/abs/quant-ph/9708022*)

[**2**] Section One of "Basic concepts in Quantum Computation", Les Houches summer school 1999, by Artur Ekert, Patrick Hayden and Hitoshi Inamori

(online at *http://arxiv.org/abs/quant-ph/0011013*)

[**3**] Makhlin Y, Schon G, Shnirman A "Nano-electronic realizations of quantum bits" J LOW TEMP PHYS vol.118 p751 (2000)

(online at *http://arxiv.org/abs/cond-mat/9912405*)

Makhlin Y, Schon G, Shnirman A "Josephson-junction qubits with controlled couplings" NATURE vol.398 p305 (1999)

(online at *http://www.nature.com/*, search for "Makhlin")

[**4**] "Quantum information processing using quantum dot spins and cavity QED" Imamoglu A, *et al* PHYSICAL REVIEW LETTERS vol.83 p4204 (1999)

(online at *http://prola.aps.org/abstract/PRL/v83/i20/p4204_1*

or *http://arxiv.org/abs/quant-ph/9904096*)

3 PROBLEMS TO SOLVE

♢ Read about spin-half in a magnetic field, in particular *spinors* and *Pauli matrices*.

Spinors: $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and **Pauli matrices:** $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Calculate the evolution of the state $u\,|\uparrow\rangle + v\,|\downarrow\rangle$ under a field in an arbitrary direction. Do this by diagonalizing the Hamiltonian and then solving Schrödinger's equation.

♢ Understand the connection between spin-half and a qubit. i.e. $|\uparrow\rangle = |0\rangle$ and $|\downarrow\rangle = |1\rangle$.

Then find the B-field that must be applied for a given time, so that it performs (i) a *phase gate* operation on the spin, (ii) a *Hadamard gate* operation on the spin.

PROJECT I : Errors due to static fields

♢ Calculate the fidelity for a spin evolving under a field $\mathbf{B} + \delta\mathbf{B}$ when the desired field is $\mathbf{B}$.

(i) assume $\delta\mathbf{B}$ is parallel to $\mathbf{B}$.

(ii) assume $\delta\mathbf{B}$ is not parallel to $\mathbf{B}$.

where necessary assume $|\delta\mathbf{B}| \ll |\mathbf{B}|$.

♢ Use above results to analyse what effect different $\delta\mathbf{B}$s have on errors in a phase gate and a Hadamard gate.

(i) If the applied field is wrong by $\delta\mathbf{B}$, which direction of $\delta\mathbf{B}$ causes the lowest fidelity?

(ii) What is the fidelity if one applies the correct field for a time δt too long?

♢ Consider what happens when the error is different for each attempt (for example $\delta\mathbf{B}$ is a random variable, taken from a Gaussian distribution). What is the average fidelity, what is the variance of the fidelity?

♠ For random $\delta\mathbf{B}$, calculate the average polarisation in the (x, y, z)-directions.

(i) Read about the *density matrix*, *pure* states and *mixed* states.

(ii) Calculate the spin's *density matrix*, ρ. Show that it is not a pure state, $\text{tr}[\rho^2] < 1$. I recommend doing this for the simplest case first, i.e. $\delta\mathbf{B}$ and $\mathbf{B}$ parallel to the z-axis.

PROJECT II : Errors due to time-dependent fields

♢ Consider a spin evolving under a field $\mathbf{B} + \delta\mathbf{B}_0 \sin(\omega t + \phi)$, where $\delta\mathbf{B}_0$ is parallel to $\mathbf{B}$.

[i] Show that if $\delta\mathbf{B}_0$ is parallel to $\mathbf{B}$ then the Hamiltonian can be diagonalised by a *time-independent* transformation.

[ii] Solve the Schrödinger equation to find the spin's evolution. Find the fidelity for this state compared with the desired state (which evolved under $\mathbf{B}$).

[iii] Assume ϕ is a random variable (for example chosen from a flat distribution) and calculate the average fidelity and its variance.

♠ Take [iii] of the above problem, and calculate the average polarisation in the (x, y, z)-directions.

(i) Read about the *density matrix*, *pure* states and *mixed* states.

(ii) Calculate the spin's *density matrix*, ρ. Show that it is not a pure state, $\text{tr}[\rho^2] < 1$. I recommend doing this first for $\mathbf{B}$ and $\delta\mathbf{B}_0$ parallel to the z-axis.

♢ Consider a spin evolving under a field $\mathbf{B} + \delta\mathbf{B}_0 \sin(\omega t + \phi)$, where $\delta\mathbf{B}_0$ is *not* parallel to $\mathbf{B}$.

[i] Show that then the Hamiltonian can only be diagonalised by a *time-dependent* transformation.

[ii] ♠ Check that this makes it hard (impossible?) to solve the Schrödinger equation.

♢ Read about time-dependent perturbation theory.

♠ Use perturbation theory to solve the spin-dynamics when $\delta\mathbf{B}_0$ is *not* parallel to $\mathbf{B}$. For this one must assume $|\delta\mathbf{B}_0| \ll |\mathbf{B}|$.

Hint: to do the calculation using 1$^{\text{st}}$ order perturbation theory (instead of 2$^{\text{nd}}$ order), one must use a trick. Never directly calculate the amplitude to stay in the nth state, $A(n \to n)$, instead get it using $|A(n \to n)|^2 = 1 - \sum_{m \neq n} |A(n \to m)|^2$.